

The first draft

SOME FRACTIONAL FUNCTIONAL INEQUALITIES AND APPLICATIONS TO SOME MINIMIZATION CONSTRAINED PROBLEMS INVOLVING A LOCAL LINEARITY

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ABSTRACT.

1. INTRODUCTION

- introduction will be added -

The fractional Laplacian is characterized as

$$\sqrt{-\Delta}^s \phi := \mathcal{F}^{-1}(|\cdot|^s \mathcal{F}(\phi)),$$

where $\hat{u} = \mathcal{F}(u)$ represents the Fourier transform of u on \mathbb{R}^n defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx,$$

if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

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2. FRACTIONAL INTEGRAL INEQUALITIES AND COMPACT EMBEDDING

In this section, we will construct the fractional Polya-Szegö inequality, and present a fractional version of Gagliardo-Nirenberg inequality. As an application, we show that the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is compactly embedded into Lebesgue spaces $L^q(\Omega)$.

2.1. Fractional Polya-Szegö inequality. We investigate the nonexpansivity of Schwarz symmetric decreasing rearrangement of functions with respect to the fractional actions $(-\Delta)^{s/2}$ for $s \geq 0$. For the basic terminology and some properties of Schwarz symmetric decreasing rearrangement, we refer Chapter 3 in [9], also [4].

Theorem 2.1. *Let $0 \leq s \leq 1$. Let u^* denote the Schwarz symmetric radial decreasing rearrangement of u . Then we have*

$$(2.1) \quad \int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u(x)|^2 dx,$$

in the sense that the finiteness of the right side implies the finiteness of the left side.

Proof. When $s = 0$, we have the equality in (2.1). We now are going to present how the kinetic energy decreases via the symmetric radial decreasing rearrangement as the differential index s increases.

To show (2.1), it is enough to prove the following:

$$(2.2) \quad \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[f^*](\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

The main idea of the proof is that the inequality (2.2) can be followed from proving the assertion: for any $\varepsilon > 0$,

$$(2.3) \quad \int_{\mathbb{R}^n} \left(\frac{|\eta|^2}{1 + \varepsilon^2 |\eta|^2} \right)^s |\mathcal{F}[u^*](\eta)|^2 d\eta \leq \int_{\mathbb{R}^n} \left(\frac{|\eta|^2}{1 + \varepsilon^2 |\eta|^2} \right)^s |\hat{u}(\eta)|^2 d\eta.$$

With change of variables $\xi = \varepsilon\eta$, (2.3) becomes

$$(2.4) \quad \frac{1}{\varepsilon^{2n}} \int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1 + |\xi|^2} \right)^s |\mathcal{F}[u^*](\xi/\varepsilon)|^2 d\xi \leq \frac{1}{\varepsilon^{2n}} \int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1 + |\xi|^2} \right)^s |\hat{u}(\xi/\varepsilon)|^2 d\xi.$$

Replace $u(x)$ by $u(x/\varepsilon)$, and we have $[u(x/\varepsilon)]^* = u^*(x/\varepsilon)$ since rearrangement commutes with uniform dilation on the space. Then (2.4) is equivalent to saying

$$(2.5) \quad \int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1 + |\xi|^2} \right)^s |\mathcal{F}[u^*](\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1 + |\xi|^2} \right)^s |\hat{u}(\xi)|^2 d\xi.$$

So it suffices to prove (2.5). Incorporating the following expression

$$\left(\frac{|\xi|^2}{1 + |\xi|^2} \right)^s = \left(1 - \frac{1}{1 + |\xi|^2} \right)^s = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \binom{s}{k} \left(\frac{1}{1 + |\xi|^2} \right)^k$$

with $\binom{s}{k} = \frac{s(s-1)\cdots(s-(k-1))}{k!}$ into each side of inequality (2.5) yields

$$\int_{\mathbb{R}^n} |\mathcal{F}[u^*](\xi)|^2 d\xi - \sum_{k=1}^{\infty} (-1)^{k+1} \binom{s}{k} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u^*](\xi)|^2 d\xi$$

and

$$\int_{\mathbb{R}^n} |\mathcal{F}[u](\xi)|^2 d\xi - \sum_{k=1}^{\infty} (-1)^{k+1} \binom{s}{k} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u](\xi)|^2 d\xi.$$

Since $(-1)^{k+1} \binom{s}{k} > 0$ with $0 < s < 1$, it remains to show that for each positive integer k

$$\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u^*](\xi)|^2 d\xi \geq \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u](\xi)|^2 d\xi.$$

We consider a Bessel kernel G_{2k} of order $2k$: $(1+|\xi|^2)^{-k} = \mathcal{F}[G_{2k}](\xi)$. Therefore with $\tilde{u}(x) = u(-x)$, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u](\xi)|^2 d\xi &= \int_{\mathbb{R}^n} \hat{G}_{2k}(\xi) \bar{\tilde{u}}(\xi) \hat{u}(\xi) d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} G_{2k}(-x) (u * \tilde{u})(x) dx \\ &= (2\pi)^n [G_{2k}(x) * (u * \tilde{u})(x)](0) \\ &= (2\pi)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} G_{2k}(y-z) \bar{u}(z) u(y) dy dz \\ (2.6) \quad &\leq (2\pi)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} G_{2k}(y-z) \bar{u}^*(z) u^*(y) dy dz \\ &= \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[u^*](\xi)|^2 d\xi. \end{aligned}$$

The *Symmetrization lemma* in [3, 9] yields the inequality (2.6) where is the only place that inequality occurs. The proof is now completed. \square

2.2. Fractional Gargliardo-Nirenberg Inequality. Gargliardo-Nirenberg inequality for fractional Laplacian is presented, and sharp form of the fractional Sobolev inequality is obtained as a corollary. Throughout this paper, C denotes various real positive constants which do not depend on functions in discussion.

Theorem 2.2. *Let $m, q, \theta \in \mathbb{R} \setminus \{0\}$ with $q \neq m\theta > 0$, $0 < s < n$, $1 < p < \frac{n}{s}$ and $1 < \frac{r}{q-m\theta}$. Then the inequality*

$$(2.7) \quad \int_{\mathbb{R}^n} |u(x)|^q dx \leq C \left(\int_{\mathbb{R}^n} \left(\sqrt{-\Delta}^s u(x) \right)^p dx \right)^{\frac{m\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r dx \right)^{\frac{q-m\theta}{r}}$$

holds for the indices with the relation

$$(2.8) \quad m\theta \left(\frac{1}{p} - \frac{s}{n} \right) + \frac{q - m\theta}{r} = 1.$$

In particular, when $m = q$, we have a fractional version of Gagliardo-Nirenberg inequality:

$$(2.9) \quad \left(\int_{\mathbb{R}^n} |u(x)|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u(x)|^p dx \right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^r dx \right)^{\frac{1-\theta}{r}}$$

for the indices with the relation

$$(2.10) \quad \theta \left(\frac{1}{p} - \frac{s}{n} \right) + \frac{1 - \theta}{r} = \frac{1}{q}.$$

Proof. For convenience, we use the notation $\|u\|_{L^t} := \left(\int_{\mathbb{R}^n} |u(x)|^t dx \right)^{\frac{1}{t}}$, and $L^t(\mathbb{R}^n)$ for any $t \in \mathbb{R} \setminus \{0\}$. First we point out that by the standard *dilation argument* the index relation (2.8) is necessary. In fact, by replacing $u(\cdot)$ with $u(\delta \cdot)$, we can observe

$$\delta^{-n} \|u\|_{L^q}^q \leq C \delta^{(s - \frac{n}{p})\theta + \frac{n(m\theta - q)}{r}} \|\sqrt{-\Delta}^s |u|^m\|_{L^p}^\theta \|u\|_{L^r}^{q - m\theta},$$

for all $\delta > 0$, which implies that $-n = (s - \frac{n}{p})\theta + \frac{n(m\theta - q)}{r}$.

Now, for any $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$(2.11) \quad \begin{aligned} \int_{\mathbb{R}^n} |u(x)|^q dx &= \int_{\mathbb{R}^n} |u(x)|^{m\theta} |u(x)|^{q - m\theta} dx \\ &\leq \| |u|^{m\theta} \|_{L^{\bar{p}}} \| |u|^{q - m\theta} \|_{L^{\bar{r}}}, \quad \frac{1}{\bar{p}} + \frac{1}{\bar{r}} = 1 \\ &= \|u\|_{L^{m\theta\bar{p}}}^{m\theta} \|u\|_{L^{(q - m\theta)\bar{r}}}^{q - m\theta}. \end{aligned}$$

We set $m\theta\bar{p} := p_0$ and $(q - m\theta)\bar{r} := r$ to have

$$(2.12) \quad \int_{\mathbb{R}^n} |u(x)|^q dx \leq \|u\|_{L^{p_0}}^{m\theta} \|u\|_{L^r}^{q - m\theta},$$

and $\frac{m\theta}{p_0} + \frac{q - m\theta}{r} = 1$. Let $\sqrt{-\Delta}^s u := f$, and we have

$$u(x) = \frac{c_{n-s}}{c_s} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-s}} dy = \frac{c_{n-s}}{c_s} \int_{\mathbb{R}^n} \frac{\sqrt{-\Delta}^s u(y)}{|x - y|^{n-s}} dy,$$

where $c_s = \frac{\Gamma(s/2)}{\pi^{s/2}}$. Indeed, we may take the Fourier transform on $\sqrt{-\Delta}^s u = f$, and take it back to have u after solving for \hat{u} . Therefore the Hardy-Littlewood-Sobolev inequality yields

$$(2.13) \quad \|u\|_{L^{p_0}} \leq \frac{c_{n-s}}{c_s} C_1 \left\| \sqrt{-\Delta}^s u \right\|_{L^p},$$

where C_1 is a positive constant (see the remark after the proof) and p satisfies

$$(2.14) \quad \frac{1}{p} + \frac{n-s}{n} = 1 + \frac{1}{p_0}.$$

This index relation combining with the index relation appeared at (2.12) implies (2.8), and (2.12) together with (2.13) implies (2.7). \square

It is known the best constant C_1 and the extremals of the Hardy-Littlewood-Sobolev inequality for some special cases (see [8] or Section 4.3 in [9]). Thanks to those cases, we have a sharp form of the fractional Sobolev inequality:

Corollary 2.3 (Fractional Sobolev inequality). *For $0 < s < n$, $1 < p < \frac{n}{s}$ and $q = \frac{pn}{n-sp}$, we have*

$$\|u\|_{L^q} \leq C_0 \left\| \sqrt{-\Delta}^s u \right\|_{L^p}.$$

The sharp constant for the inequality is

$$\pi^{s/2} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{n+s}{2})} \left\{ \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right\}^{s/n}.$$

For a special case, we emphasize the L^2 -estimate of the fractional Gagliardo-Nirenberg inequality which is applied at Section 3.

Corollary 2.4. *For $0 < s < \frac{n}{2}$, $0 < \theta < 1$ and $\theta = \frac{n(q-2)}{2qs}$, we have*

$$\|u\|_{L^q} \leq C \|\nabla_s u\|_{L^2}^\theta \|u\|_{L^2}^{1-\theta}$$

with the notation $\|\nabla_s u\|_{L^2} := \left(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx \right)^{\frac{1}{2}}$.

2.3. Fractional Rellich-Kondrachov Compactness theorem. The following theorem illustrates that the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is compactly embedded into Lebesgue spaces $L^q(\Omega)$, where Ω is bounded.

Theorem 2.5. *Let $0 < s < n$, $1 \leq p < \frac{n}{s}$ and $1 \leq q < \frac{np}{n-sp}$. Also, let $\{u_m\}$ be a sequence in $L^q(\mathbb{R}^n)$ and Ω be a bounded open set with smooth boundary. Suppose that*

$$\int_{\mathbb{R}^n} |\sqrt{-\Delta} + 1|^s u_m(x)^p dx$$

are uniformly bounded, then $\{u_m\}$ has a convergent subsequence in $L^q(\Omega)$.

Proof. Let ϕ be a smooth non-negative function with support in $\{x : |x| \leq 1\}$ and with $\int_{|x| \leq 1} \phi(x) dx = 1$. We also define $\phi^\ell(x) := \ell^n \phi(\ell x)$. By virtue of the Fractional Sobolev inequality (Corollary 2.3), it can be observed that

$$(2.15) \quad \|u_m\|_{L^q(\mathbb{R}^n)} \leq C \left\| \sqrt{-\Delta}^s u_m \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \sqrt{-\Delta} + 1^s u_m \right\|_{L^p(\mathbb{R}^n)} \leq \tilde{C}$$

for some $\tilde{C} > 0$. Hence in the spirit of Frechet-Kolmogorov theorem, it suffices to show the following (see page 50 in [11]): for any $\varepsilon > 0$ and any compact subset K of Ω , there is a constant $M > 0$ such that for $m \geq M$,

$$\|\phi^\ell * u - u\|_{L^q(K)} < \varepsilon,$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ with $\|\sqrt{-\Delta + 1}^s u\|_{L^p(\mathbb{R}^n)} \leq \tilde{C}/C$. Then using the interpolation inequality (2.12), we have

$$\|\phi^\ell * u - u\|_{L^q(K)} \leq C 2^{1-\theta} \|u\|_{L^r(\mathbb{R}^n)}^{1-\theta} \|\phi^\ell * u - u\|_{L^1(K)}^\theta,$$

with $\frac{1-\theta}{r} + \theta = \frac{1}{q}$, $r = \frac{np}{n-sp}$. Consequently, (2.15) and the fractional Gagliardo-Nirenberg inequality (Theorem 2.2) imply that

$$\|\phi^\ell * u - u\|_{L^q(K)} \leq C \|\phi^\ell * u - u\|_{L^1(K)}^\theta.$$

Now we define $f := \sqrt{-\Delta + 1}^s u$ to have $u = G_s * f$ and $\|f\|_{L^p} \leq \tilde{C}/C$, where G_s is the Bessel kernel of order s . Therefore we obtain

$$\|\phi^\ell * u - u\|_{L^1(K)} \leq C \|(\phi^\ell * G_s - G_s) * f\|_{L^p(\mathbb{R}^n)} \leq C \|\phi^\ell * G_s - G_s\|_{L^1(\mathbb{R}^n)} \rightarrow 0$$

as $m \rightarrow \infty$. \square

3. GROUND STATE SOLUTION OF FRACTIONAL SCHRÖDINGER FLOWS

We consider the following variational problem:

$$(3.1) \quad I_c = \inf \{E(u) : u \in S_c\}$$

where c is a prescribed number, $0 < s < 1$ and E is the energy functional

$$E(u) = \int_{\mathbb{R}^n} |\sqrt{-\Delta}^s u(x)|^2 dx - \int_{\mathbb{R}^n} F(|x|, u(x)) dx$$

on an admissible collection $S_c := \{u \in H^s(\mathbb{R}^n) : \int_{\mathbb{R}^n} u^2(x) dx = c^2\}$.

The aim of this work is to study the symmetry properties of minimizers of (3.1). We can also note that the solutions of (3.1) lie on the curve

$$(3.2) \quad (-\Delta)^s u + f(|x|, u) + \lambda u = 0,$$

where λ is a Lagrange multiplier and $F(r, s) = \int_0^s f(r, t) dt$. It will be interesting to study the above identity (3.2) and to find suitable assumptions on f for which all the solutions of (3.2) are radial and radially decreasing. Note that for the classical Laplacian, H. JeanJean(?) and C. Stuart have completely solved the problem. It is also worth to study a fractional Schrödinger equation

$$(3.3) \quad \begin{cases} i\partial_t \Phi + (-\Delta)^s \Phi + f(|x|, \Phi) = 0 \\ \Phi(x, 0) = \Phi_0(x) \end{cases}$$

for which ground state solutions u of (3.2) give rise to ground state solitary wave Φ of (3.3). The minimizing problem (3.1) that we are going to look at imposes the following assumptions on the function F :

(F₀) $F : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, that is to say:

- $F(\cdot, s) : [0, \infty) \rightarrow \mathbb{R}$ is measurable for all $s \in \mathbb{R}$ and
 - $F(r, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for almost every $r \in [0, \infty)$.
- (F₁) $F(r, s) \leq F(r, |s|)$ for almost every $r \geq 0$ and all $s \in \mathbb{R}$.
(F₂) There are $K > 0$ and $0 < l < \frac{4s}{n}$ satisfying for any $r, s \geq 0$,

$$0 \leq F(r, s) \leq K(s^2 + s^{l+2}).$$

(F₃) For every $\varepsilon > 0$, there exist $R_0, s_0 > 0$ such that $F(r, s) \leq \varepsilon|s|^2$ for almost every $r \geq R_0$ and all $0 \leq s < s_0$.

(F₄) The mapping $(t, y) \mapsto F(\frac{1}{t}, y)$ is super-modular on $\mathbb{R}_+ \times \mathbb{R}_+$, in other words,

$$F(r, a) + F(R, A) \geq F(r, A) + F(R, a)$$

for all $r < R$ and $a < A$.

Theorem 3.1. *Under the conditions (F0) \sim (F4), the minimizing problem (3.1) admits a Schwarz symmetric minimizer for any fixed constant c . Moreover if (F4) holds with a strict sign, then for any c , all minimizers of (3.1) are Schwarz symmetric.*

A Schwarz symmetric function is a radial decreasing function. For more detailed accounts, we refer [BH].

Proof. 1. *Well-posedness of the problem (3.1)* (that is, $I_c > -\infty$): We first show that all minimizing sequences are bounded in $H^s(\mathbb{R}^n)$. By (F1) and (F2), we can write

$$\int F(|x|, u(x)) dx \leq \int F(|x|, |u(x)|) dx \leq Kc^2 + K \int |u(x)|^{l+2} dx.$$

By virtue of the fractional Gagliardo-Nirenberg inequality (Corollary 2.4) and Young's inequality, there exists constant K' such that

$$\begin{aligned} (3.4) \quad \int_{\mathbb{R}^n} |u(x)|^{l+2} dx &\leq K' \left\{ \int_{\mathbb{R}^n} u^2(x) dx \right\}^{(1-\theta)\frac{(l+2)}{2}} \|\nabla_s u\|_{L^2}^{\theta(l+2)}. \\ &\leq K' \frac{\varepsilon^p}{p} \left\{ \|\nabla_s u\|_{L^2}^2 \right\}^{p\theta\frac{(l+2)}{2}} + \frac{K'}{q\varepsilon^q} \left\{ \int_{\mathbb{R}^n} u^2(x) dx \right\}^{q(1-\theta)\frac{(l+2)}{2}} \end{aligned}$$

for any $\varepsilon > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\theta = \frac{nl}{2s(l+2)}$. We choose $p = \frac{2}{\theta(l+2)} = \frac{4s}{nl}$ to get

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^{l+2} dx &\leq \frac{K'}{p} \varepsilon^p \left\{ \|\nabla_s u\|_{L^2}^2 \right\} + \frac{K'}{q\varepsilon^q} \left\{ \int_{\mathbb{R}^n} u^2(x) dx \right\}^{q(1-\theta)\frac{l+2}{2}} \\ &= \frac{K'}{p} \varepsilon^p \|\nabla_s u\|_{L^2}^2 + \frac{K'}{q\varepsilon^q} c^{q(1-\theta)(l+2)}. \end{aligned}$$

Therefore applying (F2), we conclude

$$\begin{aligned} E(u) &\geq \frac{1}{2} \|\nabla_s u\|_{L^2}^2 - Kc^2 - \frac{K'K}{p} \varepsilon^p \|\nabla_s u\|_{L^2}^2 - \frac{K'K}{q\varepsilon^q} c^{q(1-\theta)(l+2)} \\ &= \left(\frac{1}{2} - \frac{K'k}{p} \varepsilon^p \right) \|\nabla_s u\|_{L^2}^2 - Kc^2 - \frac{K'K}{q\varepsilon^q} c^{q(1-\theta)(l+2)}. \end{aligned}$$

Remark 3.2. 1. If we allow $l = \frac{4s}{n}$ in (F2), the problem (3.1) still makes sense for sufficiently small values of c . In fact, with $\theta = \frac{2}{l+2}$ and in view of (3.4) we have

$$\int_{\mathbb{R}^n} |u(x)|^{l+2} dx \leq K' c^{\frac{4s}{n}} \|\nabla_s u\|_{L^2}^2$$

for $u \in S_c$. Hence we get

$$\begin{aligned} E(u) &\geq \frac{1}{2} \|\nabla_s u\|_{L^2}^2 - Kc^2 - K'K c^{\frac{4s}{n}} \|\nabla_s u\|_{L^2}^2 \\ &= \left(\frac{1}{2} - K'K c^{\frac{4s}{n}} \right) \|\nabla_s u\|_{L^2}^2 - Kc^2. \end{aligned}$$

Thus $I_c > -\infty$ and all minimizing sequences are bounded in $H^s(\mathbb{R}^n)$ provided that $0 < c < (\frac{1}{2KK'})^{\frac{n}{4s}}$.

2. We can prove that $I_c = -\infty$ for $l > \frac{4s}{n}$.

2. *Existence of a Schwarz symmetric minimizing sequence.* First note that if $u \in H^s(\mathbb{R}^n)$, then $|u| \in H^s(\mathbb{R}^n)$. In view of (F1), we certainly have that

$$E(|u|) \leq E(u), \quad \text{for all } u \in H^s(\mathbb{R}^n).$$

Now by virtue of the fractional Polya-Szegö inequality (Theorem 2.1):

$$\|\nabla_s |u|^*\|_{L^2} \leq \|\nabla_s |u|\|_{L^2}$$

and Theorem 1 of [BH], we can observe that

$$\int_{\mathbb{R}^n} F(|x|, |u|(x)) dx \leq \int_{\mathbb{R}^n} F(|x|, |u|^*(x)) dx.$$

Thus, without loss of generality, we may say that (3.1) always admits a Schwarz symmetric minimizing sequence.

3. Let $\{u_m\} = \{u_m^*\}$ be a Schwarz symmetric minimizing sequence. If $\{u_m\}$ converges weakly to u in $H^s(\mathbb{R}^n)$, then

$$E(u) \leq \liminf_{m \rightarrow \infty} E\{u_m\}.$$

The weak lower semi-continuity of L^2 -norm yields

$$\|\nabla_s u\|_{L^2} \leq \liminf_{m \rightarrow \infty} \|\nabla_s u_m\|_{L^2}.$$

Hence the assertion will follow by showing that

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} F(|x|, u_m(x)) dx = \int_{\mathbb{R}^n} F(|x|, u(x)) dx.$$

For $R > 0$, let us first prove that:

$$\lim_{m \rightarrow \infty} \int_{|x| \leq R} F(|x|, u_m(x)) dx = \int_{|x| \leq R} F(|x|, u(x)) dx.$$

By the fractional Rellich-Kondrachov theorem (Theorem 2.5), $\{u_m\}$ converges strongly to u in $L^{l+2}(\{x : |x| \leq R\})$. Thus there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ such that $u_{m_k}(x) \rightarrow u(x)$ for almost every $|x| \leq R$ and there is $h \in L^{l+2}(\{x : |x| \leq R\})$ satisfying $|u_{m_k}| \leq h$. We apply (F2) to have

$$F(|x|, u_{m_k}(x)) \leq K(h^2(x) + h^{l+2}(x)).$$

Noticing that $h^2 + h^{l+2} \in L^1(\{x : |x| \leq R\})$, the dominated convergence theorem gives

$$\lim_{m \rightarrow \infty} \int_{|x| \leq R} F(|x|, u_m(x)) dx = \int_{|x| \leq R} F(|x|, u(x)) dx.$$

Since $u_m = u_m^*$, we now have

$$\omega_n |x|^n u_m^2(x) \leq \int_{|y| \leq |x|} u_m^2(y) dy \leq c^2,$$

where ω_n is the measure of the n -dimensional unit ball. Thus we get

$$u_m(x) \leq \frac{c}{\omega_n^{\frac{n}{2}} |x|^{\frac{n}{2}}} \leq \frac{c}{\omega_n^{\frac{n}{2}} R^{\frac{n}{2}}}, \quad \text{for all } |x| > R.$$

Therefore for $\varepsilon > 0$ and R sufficiently large, we obtain by using (F3) that

$$\int_{|x| > R} F(|x|, u_m(x)) dx \leq \varepsilon \int_{|x| > R} u_m^2(x) dx < \varepsilon c^2,$$

which in turn implies that $\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{|x| > R} F(|x|, u_m(x)) dx = 0$. Since u inherits all the properties used to get the above limit, it follows also that

$$\lim_{R \rightarrow \infty} \int_{|x| > R} F(|x|, u(x)) dx = 0.$$

4. We claim that $u \in S_c$. Notice that $S_c = H^s(\mathbb{R}^n) \cap \Lambda^{-1}(\{c\})$, where Λ is defined by $\Lambda(u) := \|u\|_{L^2}$ for $u \in L^2(\mathbb{R}^n)$. We choose a Schwarz symmetric minimizing sequence $\{u_m\} \subset S_c$ converging weakly to u in $H^s(\mathbb{R}^n)$, and so it converges strongly to u in $L^2(\mathbb{R}^n)$. Hence we have that $u \in H^s(\mathbb{R}^n)$ and $u \in \Lambda^{-1}(\{c\})$. Indeed, since Λ is continuous, $\Lambda^{-1}(\{c\})$ is a closed set in $L^2(\mathbb{R}^n)$.

- References should be added and replaced -

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